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DERIVATION OF A TWO FUNCTION EXPRESSION FOR DISPLACEMENTS IN TORSION-FREE AXI-SYMMETRIC DEFORMATIONS, AND ITS RELATIONS WITH OTHER SOLUTIONS

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Abstract—Several forms of general solution for torsion-free axi-symmetric deformation have been proposed to the Lamé equation of elastic equilibrium. Most of these, excepting Love's biharmonic solution are generic to the Papkovich solution in terms of four harmonic functions for the general three dimensional case. In this work a two harmonic function form for the axi-symmetric case is proposed which is different to that of Boussinesq.

Clearly the several forms of solution are related and this paper sets out details of these relationships and equivalences in such a way that the consequences of the varying forms and their completeness is more fully understood.

1. INTRODUCTION

The solution given by Papkovich[1] to Lamé's equation for the equilibrium of a three dimensional elastic body in the Cartesian coordinate system (x, y, z) is written as

$$U_x = \alpha\Phi_x - \frac{\partial}{\partial x}(x\Phi_x + y\Phi_y + z\Phi_z + \Phi_0),$$

$$U_y = \alpha\Phi_y - \frac{\partial}{\partial y}(x\Phi_x + y\Phi_y + z\Phi_z + \Phi_0),$$

$$U_z = \alpha\Phi_z - \frac{\partial}{\partial z}(x\Phi_x + y\Phi_y + z\Phi_z + \Phi_0), \tag{1.1}$$

where (U_x, U_y, U_z) is the displacement vector, α denotes $4(1-\nu)$ with ν being the Poisson ratio, and Φ_x, Φ_y, Φ_z, Φ_0 are four harmonic functions.

When this solution is applied to the case of torsion-free axi-symmetric deformations, with z being the axis of symmetry, Φ_x and Φ_y can be omitted, as shown by [2], giving the well known Boussinesq solution[3], with Φ_0, Φ_z being functions of only r and z. This solution is widely used in elasticity theory due to its generality, completeness and simplicity.

On the other hand, when the conditions for the omission of Φ_z are satisfied (see [2]), the solution for torsion-free axi-symmetric deformations assumes the following form.

$$u(r, z) = \alpha A(r, z, \theta) - \frac{\partial}{\partial r}[rA(r, z, \theta) + B(r, z, \theta)], \tag{1.2a}$$

$$v(r, z) = \alpha C(r, z, \theta) - \frac{1}{r}\frac{\partial}{\partial \theta}[rA(r, z, \theta) + B(r, z, \theta)] \equiv 0, \tag{1.2b}$$

$$w(r, z) = -\frac{\partial}{\partial z}[rA(r, z, \theta) + B(r, z, \theta)], \tag{1.2c}$$

where $B(r, z, \theta)$ is the cylindrical form of $\Phi_0(x, y, z)$; $A(r, z, \theta)$ and $C(r, z, \theta)$ are given by

$$\Phi_x(x, y, z) + i\Phi_y(x, y, z) = [A(r, z, \theta) + iC(r, z, \theta)]\, e^{i\theta}, \tag{1.3}$$

and (u, v, w) is the displacement vector in cylindrical coordinates.

The difficulty involved in the use of this form of solution is the dependence of A, B and C on θ, a point not noted by previous authors on this subject.

In this paper it will be proved that there exist other functions $a_0(r, z)$ and $b_0(r, z)$ such that

$$u(r, z) = \alpha a_0(r, z) - \frac{\partial}{\partial r}[ra_0(r, z) + b_0(r, z)], \tag{1.4a}$$

$$v(r, z) = 0, \tag{1.4b}$$

$$w(r, z) = -\frac{\partial}{\partial z}[ra_0(r, z) + b_0(r, z)], \tag{1.4c}$$

$$\nabla^2 a_0(r, z) = \frac{a_0(r, z)}{r^2}, \tag{1.5a}$$

$$\nabla^2 b_0(r, z) = 0. \tag{1.5b}$$

This solution is known as an alternative form of Boussinesq–Papkovich solution to the torsion-free axi-symmetric problem, and has been used by Lur'e in his book[5]. But no convincing derivation of this solution from the general solution has not previously been systematically incorporated into Papkovich's general solution.

In the second part of this paper, the above solution for displacements is proved, under appropriate conditions, to be equivalent to the well known Boussinesq solution. A variation of the present solution is also mentioned, this variation expresses displacements in terms of two harmonic functions, and is very similar to the solution used by Sadowsky in [6]. To complete the cycle of inter-relationship between various solutions to the Lamé equations the connection between the Boussinesq harmonic function form and Love's biharmonic form is subsequently presented.

Thus, to the end of this paper, the inter-connections between various solutions to the problem of torsion-free axi-symmetric deformations are apparent. Hence the choice for one of these methods to a particular application in a boundary value problem depends only on its convenience and completeness relative to the others.

2. DISPLACEMENTS IN TERMS OF TWO FUNCTIONS, INDEPENDENT OF θ

The application of Papkovich solution of torsion-free axi-symmetric deformations gives rise to (1.2), with $A(r, z, \theta)$ and $C(r, z, \theta)$ defined by (1.3).

Since Φ_x and Φ_y are harmonic, A and C must satisfy

$$\nabla^2(A + iC)\, e^{i\theta} = 0,$$

or

$$\nabla^2 A - \frac{A}{r^2} - \frac{2}{r^2}\frac{\partial C}{\partial \theta} = 0, \qquad \nabla^2 C - \frac{C}{r^2} + \frac{2}{r^2}\frac{\partial A}{\partial \theta} = 0.$$

Using (1.2), the above two equations, together with the equation for $B(r, z, \theta)$, can be written as,

$$\nabla^2 A(r, z, \theta) = \frac{A}{r^2} + \frac{2}{\alpha r^3}\frac{\partial^2}{\partial \theta^2}(rA + B), \tag{2.1a}$$

$$\nabla^2 B(r, z, \theta) = 0, \tag{2.1b}$$

$$\nabla^2 C(r, z, \theta) = -\frac{C}{r^2} - \frac{2}{r}\frac{\partial C}{\partial r}. \tag{2.1c}$$

Using (1.2c), (1.2a) and (1.2b), in this order, it is straightforward to prove that

$$\frac{\partial}{\partial \theta}(rA + B) = \text{function of } (r, \theta),$$

$$\frac{\partial A}{\partial \theta} = \text{function of } (r, \theta),$$

$$rC = \text{function of } (r, \theta).$$

And, as a result,

$$\frac{\partial B}{\partial \theta} = \text{function of } (r, \theta).$$

Hence, A, B and C can be rewritten as

$$A(r, z, \theta) = a(r, \theta) + a_1(r, z), \tag{2.2a}$$

$$B(r, z, \theta) = b(r, \theta) + b_2(r, z), \tag{2.2b}$$

$$C(r, z, \theta) = \frac{1}{r} c(r, \theta). \tag{2.2c}$$

By the use of (1.2) and (2.2) it can be shown that

$$\frac{\partial c(r, \theta)}{\partial r} = \frac{\partial a(r, \theta)}{\partial \theta}.$$

This equation establishes the existence of a potential function $\phi(r, \theta)$ defined by

$$\phi(r, \theta) = \int_{(r_0, \theta_0)}^{(r, \theta)} [a(r, \theta)\, \mathrm{d}r + c(r, \theta)\, \mathrm{d}\theta],$$

where r_0, θ_0 are arbitrary reference values, $\phi(r, \theta)$ has the property of

$$\frac{\partial \phi(r, \theta)}{\partial r} = a(r, \theta),$$

and

$$\frac{\partial \phi(r, \theta)}{\partial \theta} = c(r, \theta).$$

With this newly defined function ϕ, the functions A and C can be rewritten as

$$A(r, z, \theta) = \frac{\partial \phi(r, \theta)}{\partial r} + a_1(r, z), \tag{2.3a}$$

$$C(r, z, \theta) = \frac{1}{r} \frac{\partial \phi(r, \theta)}{\partial \theta}. \tag{2.3c}$$

Using (2.3a), (2.3b) and (2.2b), $v(r, z)$ can now be written as

$$v(r, z) = \frac{\alpha}{r} \frac{\partial \phi(r, \theta)}{\partial \theta} - \frac{1}{r} \frac{\partial}{\partial \theta} \left(r \frac{\partial \phi(r, \theta)}{\partial r} + b(r, \theta) \right) \equiv 0.$$

This equation allows $b(r, \theta)$ to be related to $\phi(r, \theta)$ by

$$\alpha \phi - r \frac{\partial \phi}{\partial r} - b = f(r),$$

where $f(r)$ is a function of r only. Thus $B(r, z, \theta)$ can now be written as

$$B(r, z, \theta) = \alpha \phi(r, \theta) - r \frac{\partial \phi(r, \theta)}{\partial r} - f(r) + b_2(r, z),$$

or

$$B(r, z, \theta) = \alpha\phi(r, \theta) - r\frac{\partial\phi(r, \theta)}{\partial r} + b_1(r, z), \tag{2.3b}$$

with $b_1(r, z)$ equal to the sum of $f(r)$ and $b_2(r, z)$.

Equations (2.3) allow displacements to be written as,

$$u(r, z) = \alpha a_1(r, z) - \frac{\partial}{\partial r}(r a_1(r, z) + b_1(r, z)) \tag{2.4a}$$

$$v(r, z) = 0 \tag{2.4b}$$

$$w(r, z) = -\frac{\partial}{\partial z}(r a_1(r, z) + b_1(r, z)). \tag{2.4c}$$

It should be noted that no attempt has been made to demonstrate that $a_1(r, z)$ and $b_1(r, z)$ are harmonic.

3. DISPLACEMENTS IN TERMS OF ONE HARMONIC AND ONE HARMONIC RELATED FUNCTION

Equations (2.1c) and (2.3c) give,

$$\nabla^2\left(\frac{1}{r}\frac{\partial\phi(r, \theta)}{\theta}\right) + \frac{1}{r^3}\frac{\partial\phi(r, \theta)}{\partial\theta} + \frac{2}{r}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\phi(r, \theta)}{\partial\theta}\right) = 0,$$

or

$$\frac{\partial}{\partial\theta}\nabla^2\phi(r, \theta) = 0.$$

This implies

$$\nabla^2\phi(r, \theta) = h(r),$$

where $h(r)$ is a function of only r.

Another function $\psi(r, \theta)$, which is harmonic, can be defined by

$$\psi(r, \theta) = \phi(r, \theta) - \int_{r_0}^{r}\frac{1}{r}\int_{r_0}^{r} r h(r)\,\mathrm{d}r\,\mathrm{d}r. \tag{3.1}$$

Hence, A, B and C can be rewritten as

$$A(r, z, \theta) = \frac{\partial\psi(r, \theta)}{\partial r} + a_0(r, z), \tag{3.2a}$$

$$B(r, z, \theta) = \alpha\psi(r, \theta) - r\frac{\partial\psi(r, \theta)}{\partial r} + b_0(r, z), \tag{3.2b}$$

$$C(r, z, \theta) = \frac{1}{r}\frac{\partial\psi(r, \theta)}{\partial\theta}, \tag{3.2c}$$

where the functions $a_0(r, z)$, $b_0(r, z)$ are given by

$$a_0(r, z) = \frac{1}{r}\int_{r_0}^{r} r h(r)\,\mathrm{d}r + a_1(r, z)$$

and

$$b_0(r, z) = \alpha\int_{r_0}^{r}\frac{1}{r}\int_{r_0}^{r} r h(r)\,\mathrm{d}r - \int_{r_0}^{r} r h(r)\,\mathrm{d}r + b_1(r, z).$$

Displacements can now be written in the desired form (1.4).

Using (2.1a), (2.1b) together with (3.2c), (3.2b) it can be shown that

$$\nabla^2 a_0 - \frac{a_0}{r^2} + \left(\frac{\partial}{\partial r}\alpha^2\psi\right) = 0.$$

$$\nabla^2 b_0 + \left[\alpha\nabla^2\psi - \frac{1}{r}\frac{\partial}{\partial r}(r^2\nabla^2\psi)\right] = 0.$$

Knowing that ψ is harmonic, the above equations can be written in the form of (1.5). Thus it can be now stated that:

The displacements in torsion-free axi-symmetric deformations are expressible by (1.4) *with* $a_0(r, z)$ *and* $b_0(r, z)$ *satisfying* (1.5), *when the conditions for the omission of* Φ_z (*see* [2]) *in Papkovich's general solution are satisfied.*

4. ALTERNATIVE FORM OF (1.4) AND (1.5)

If displacements are given by the eqns (1.4) and (1.5), then a function $d(r, z)$ can be defined by

$$d(r, z) = \int_{r_1}^{r} a_0(r, z)\,\mathrm{d}r,$$

where r_1 is an arbitrary reference value. Hence

$$\frac{\partial}{\partial r}\nabla^2 d(r, z) = \nabla^2\frac{\partial d}{\partial r} - \frac{1}{r^2}\frac{\partial d}{\partial r}$$

$$= \nabla^2 a_0 - \frac{a_0}{r^2} = 0.$$

This leads to

$$\nabla^2 d(r, z) = p(z).$$

Define the function $d_0(r, z)$ by

$$d_0(r, z) = d(r, z) - \int_{z_1}^{z}\int_{z_1}^{z} p(z)\,\mathrm{d}z\,\mathrm{d}z,$$

where z_1 is an arbitrary reference value.

Then

$$u = -\frac{\partial}{\partial r}\left(-\alpha d_0 + r\frac{\partial d_0}{\partial r} + b_0\right), \tag{4.1}$$

$$w = -\frac{\partial}{\partial z}\left(r\frac{\partial d_0}{\partial r} + b_0\right),$$

with

$$\nabla^2 d_0 = 0,$$

$$\nabla^2 b_0 = 0. \tag{4.2}$$

On the other hand, if displacements are given by (4.1) and (4.2), by defining

$$a_0(r, z) = \frac{\partial}{\partial r} d_0(r, z),$$

then displacements are also expressible by (1.4) and (1.5).

Thus (4.1) and (4.2) are the equivalent representation of (1.4) and (1.5).

A trivial variation of (4.1) and (4.2) gives the following form

$$u = \frac{\partial}{\partial r}\left(r\frac{\partial\phi_1}{\partial r} + \psi_1\right), \qquad w = \frac{\partial}{\partial r}\left[4(1-\nu)\phi_1 + r\frac{\partial\phi_1}{\partial r} + \psi_1\right],$$

with

$$\nabla^2\psi_1 = 0, \qquad \nabla^2\phi_1 = 0.$$

This latter form is thought to have been used by Sadowsky in his unpublished lecture notes[6].

5. RELATION BETWEEN (4.1) AND (4.2) AND BOUSSINESQ SOLUTION

Assuming that displacements are given by (4.1) and (4.2), define $D(r, z)$ and $E(r, z)$ by

$$D(r, z) = r\frac{\partial d_0}{\partial r} + z\frac{\partial d_0}{\partial z} + b_0 - \alpha d_0,$$

$$E(r, z) = -\frac{\partial d_0}{\partial z}.$$

Then, by the use of the above equations together with (4.1), displacements can be written as

$$u(r, z) = -\frac{\partial}{\partial r}(zE + D),$$

$$w(r, z) = \alpha E - \frac{\partial}{\partial z}(zE + D). \tag{5.1}$$

Using (4.2), it is only routine calculation to show that D and E are harmonic, i.e.

$$\nabla^2 D(r, z) = 0. \tag{5.2a}$$

$$\nabla^2 E(r, z) = 0. \tag{5.2b}$$

On the other hand, if displacements are given by (5.1) and (5.2) then by defining

$$d(r, z) = -\int_{z_1}^{z} E(r, z)\,\mathrm{d}z,$$

it can be shown that,

$$\frac{\partial}{\partial z}\nabla^2 d(r, z) = 0, \text{ by virtue of (5.2b)},$$

or

$$\nabla^2 d(r, z) = q(r).$$

Define another function

$$d_0(r, z) = d(r, z) - \int_{r_1}^{r}\frac{1}{r}\int_{r_1}^{r} rq(r)\,\mathrm{d}r\,\mathrm{d}r$$

which is harmonic, and another one

$$b_0(r, z) = D(r, z) + \alpha d_0 + r\frac{\partial d_0}{\partial r} + z\frac{\partial d_0}{\partial z}$$

which is also harmonic by (5.2) and the harmonicity of d_0.

Using (5.1), it is trivial to show that displacements are also expressible by (4.1), and since d_0 and b_0 are harmonic, (4.2) is satisfied.

6. RELATION BETWEEN BOUSSINESQ SOLUTION AND LOVE SOLUTION

Assuming that displacements are given by (5.1) and (5.2), define χ_0 by

$$\chi_0(r, z) = \int_{z_1}^{z} (zE + D)\, dz,$$

then it can be shown that

$$\frac{\partial}{\partial z}\nabla^2\chi_0 = 2\frac{\partial E}{\partial z}.$$

The above equation implies

$$\nabla^2\chi_0 = 2E + s(r),$$

where $s(r)$ is a function of only r.

Define another function,

$$\chi = \chi_0 - \int_{r_1}^{r}\frac{1}{r}\int_{r_1}^{r} rs(r)\, dr\, dr.$$

Then χ has the property of

$$\frac{\partial \chi}{\partial z} = zE + D,$$

and

$$\nabla^2\chi = 2E.$$

Hence displacements are given, using the above two equations in conjunction with (5.1), by

$$u = -\frac{\partial^2\chi}{\partial r\partial z},$$

$$w = 2(1-\nu)\nabla^2\chi - \frac{\partial^2\chi}{\partial z^2}. \quad (6.1)$$

Since E is harmonic, the equation relating χ and E gives

$$\nabla^4\chi = 0. \quad (6.2)$$

On the other hand, when displacements are given by (6.1) and (6.2), the functions D and E defined by

$$E = \frac{1}{2}\nabla^2\chi, \qquad D = \frac{\partial\chi}{\partial z} - \frac{1}{2}z\nabla^2\chi,$$

can obviously satisfy (5.1). They are also harmonic since χ is biharmonic.

7. RELATIVE USEFULNESS OF DIFFERENT APPROACHES

The solution presented in this paper has been successfully used by Lur'e[5] for the axi-symmetric problems of a cylinder. Boussinesq's solution has also been used in ([5], p. 391); this solution has an advantage of having z multiplying one harmonic function, an example where this is used to its advantage can be found in [9]. An alternative form given by Sadowski has been fruitful in the variational approach to the end problem of a cylinder[7]. Also the biharmonic (stress) function derived by Love is familiar and is adopted in standard texts, such as [10]. The use of Hankle transforms to reduce the axisymmetric form of the biharmonic equations to an ordinary differential equation in the axial dimension is extensively discussed in Sneddon[11].

A simple guide in choosing a particular solution for a given probelm can be expressed as; (a) Compatibility conditions being satisfied. (All the methods considered above do meet this requirement.) (b) Simplicity of the form of the solution. (c) Separation of a dimension from the rest. (d) Ease of adopting the solution form to a particular method (as illustrated in [7]).

Further discussion on the relative values of different methods can be found in Klemm and Little[8]. A full discussion of the choice of these methods would be very involved and is outside the scope of this paper.

8. CONCLUSIONS

When the conditions for the omission of Φ_z (see Ref. [2]) are satisfied displacements in torsion-free axi-symmetric problems are expressible in terms of two functions $a_0(r, z)$ and $b_0(r, z)$ in the manner specified by (1.4) and (1.5). The conditions for this representation are thus more restricted than the corresponding ones for Boussinesq representation.

The various representations of [(1.4) and (1.5)], of [(4.1) and (4.2], of Boussinesq and of Love are all interconnected. When the transformation between any two of the forms is possible, these two forms are equivalent and if one is complete, so is the other. Thus for displacements which are expressible in more than one of the equivalent representations, the choice of one over the others depends only on its convenience.

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